Multi-Objective Simplex Method Algorithm

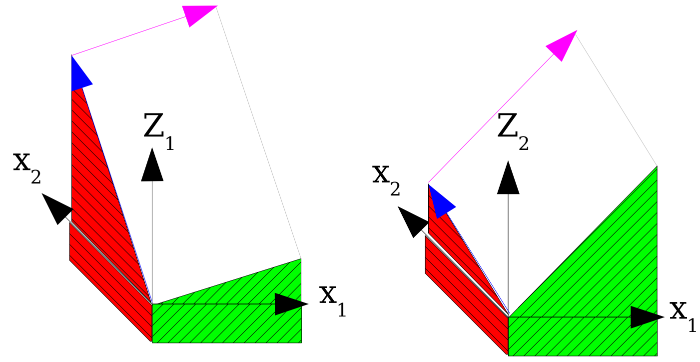
Purpose

Maximize

multiple linear objectives

subject to

multiple linear constraints



Definitions

Variables: $x_1, x_2, \ldots, x_n$

Basic variable #2: x_{B2}

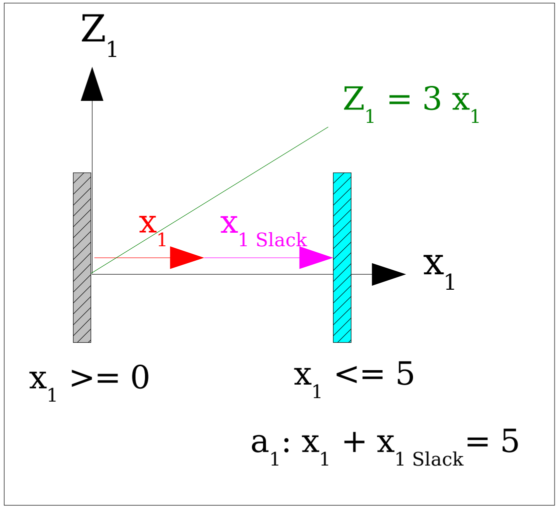
Non-basic variable #4: x_{NB4}

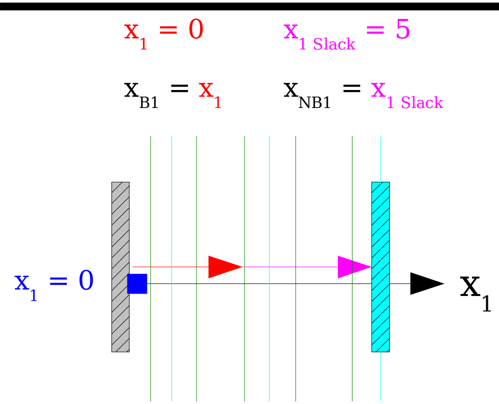
Objectives: $Z_1, Z_2, \ldots, Z_k$

Constraints: $a_1, a_2, \ldots, a_m$

Reduced cost gradients: $f_1, f_2, \ldots, f_m$

One Dimensional (1D) Example





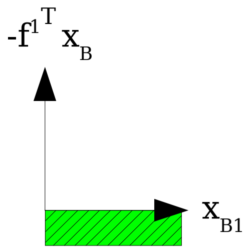
$$Variables^T = [x_1 \, x_{1\,Slack}]$$

$$z^T = c^T = [3 \, 0]$$

$$x_B^T = [x_{1\,Slack}]$$

$$c_B^T = [0]$$

$$\frac{da_1}{dx_{B1}} = \frac{da_1}{dx_{1\,Slack}} = 1$$



$$-f^1_{x_B} = z_B - c_B \cdot \frac{da_1}{dx_{B1}}$$

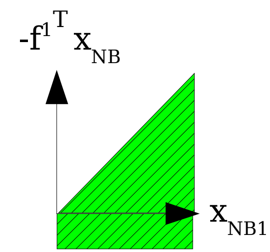
$$= 0 - 0 \cdot 1 = 0$$

$$x = \begin{Bmatrix} x_1 \\ x_{1\,Slack} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 5 \end{Bmatrix}$$

$$x_{NB}^T = [x_1]$$

$$c_{NB}^T = [3]$$

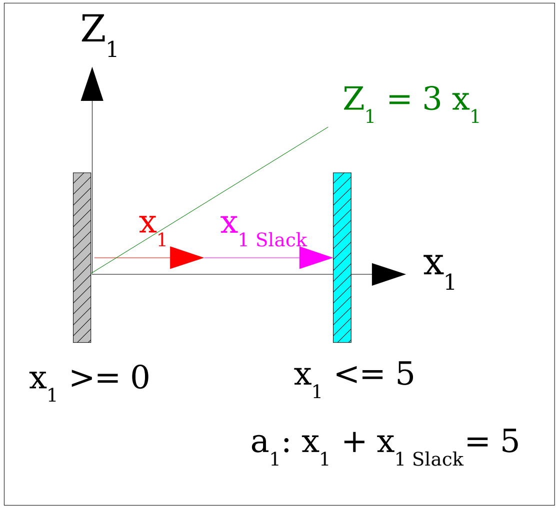
$$\frac{da_1}{dx_{NB1}} = \frac{da_1}{dx_1} = 1$$

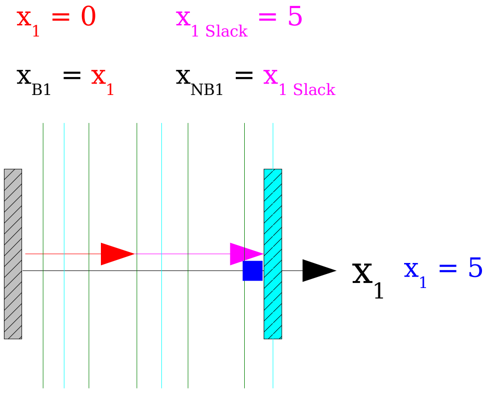


$$-f^1_{x_{NB}} = z_{NB} - c_B \cdot \frac{da_1}{dx_{NB1}}$$

$$= 3 - 0 \cdot 1 = 3$$

One Dimensional (1D) Example

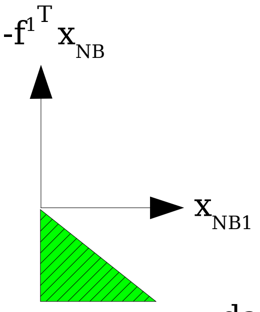




$$x_{NB}^T=[x_1]$$

$$c_{NB}^T=[0]$$

$$\frac{da_1}{dx_{NB1}}=\frac{da_1}{dx_1}=1$$



$$-f^1_{x_{NB}}=z_{NB}-c_B\cdot\frac{da_1}{dx_{NB1}}$$

$$=0-3\cdot 1=-3$$

$$\text{Variables}^T=[x_1\ x_{1\,Slack}]$$

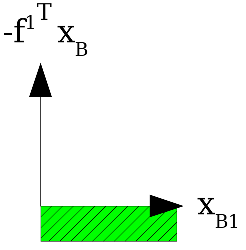
$$z^T=c^T=[3\ 0]$$

$$x=\begin{Bmatrix} x_1 \\ x_{1\,Slack} \end{Bmatrix}=\begin{Bmatrix} 5 \\ 0 \end{Bmatrix}$$

$$x_B^T=[x_1]$$

$$c_B^T=[3]$$

$$\frac{da_1}{dx_{B1}}=\frac{da_1}{dx_{1\,Slack}}=1$$



$$-f^1_{x_B}=z_B-c_B\cdot\frac{da_1}{dx_{B1}}$$

$$=3-3\cdot 1=0$$

Zeleny Multi-Objective Simplex Method

Move from one constrained point to another while trying to maximize all objectives.
(Detailed steps in Cohon and Lecture Notes)

Step 4
Is the current solution obviously noninferior?

Step 5
Is the current solution obviously inferior?

Step 6
Is the current solution noninferior by a less obvious manner?

Step 8
Is one search direction obviously better than all others? Even if it leads to a decrease in any of the objectives?

Step 9a
Do any of the search directions lead to a change in the objectives? Even decreases in the objectives are allowed.
If so, it may lead to unexplored bases (to be checked in Step 12)

Step 11
Find an unexplored bases (with a new nonbasic variable) in storage

Step 12
Would the introduction of a nonbasic variable lead to an unexplored basis?

Step 13
Introduce a nonbasic variable and remove a basic variable to form a new basis

Step 4a

Is the current solution obviously noninferior?

(Are the reduced gradients all non-negative for a particular objective?)

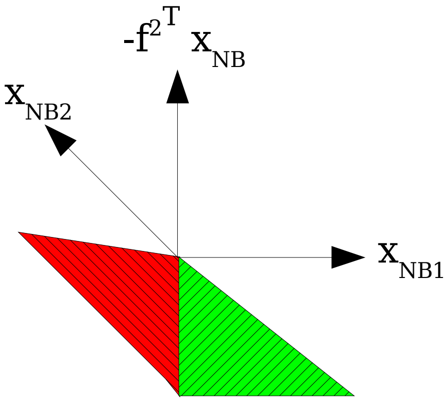
Not noninferior

Noninferior

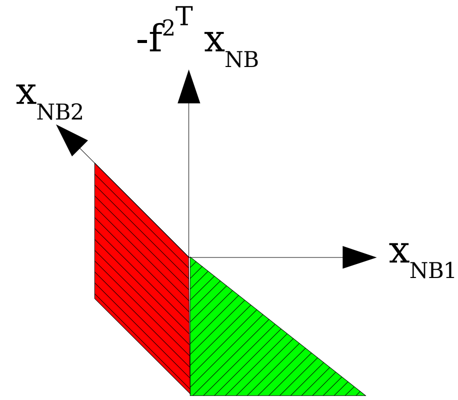
Step 4b and 4c

Is the current solution uniquely noninferior?

(Do any of the non-basic variables have reduced cost equal to zero?)



Uniquely Noninferior



Not Uniquely Noninferior

Step 4C

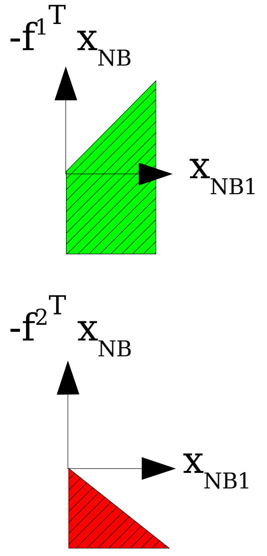
Find the other points that this leads to

Step 5

Is the current solution obviously inferior?

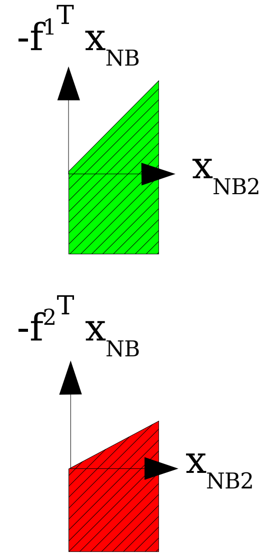
(Will the introduction of a nonbasic variable lead to an increase in all objectives?)

Variable x_{NB1}



Not clearly inferior

Variable x_{NB2}



Current solution is clearly inferior

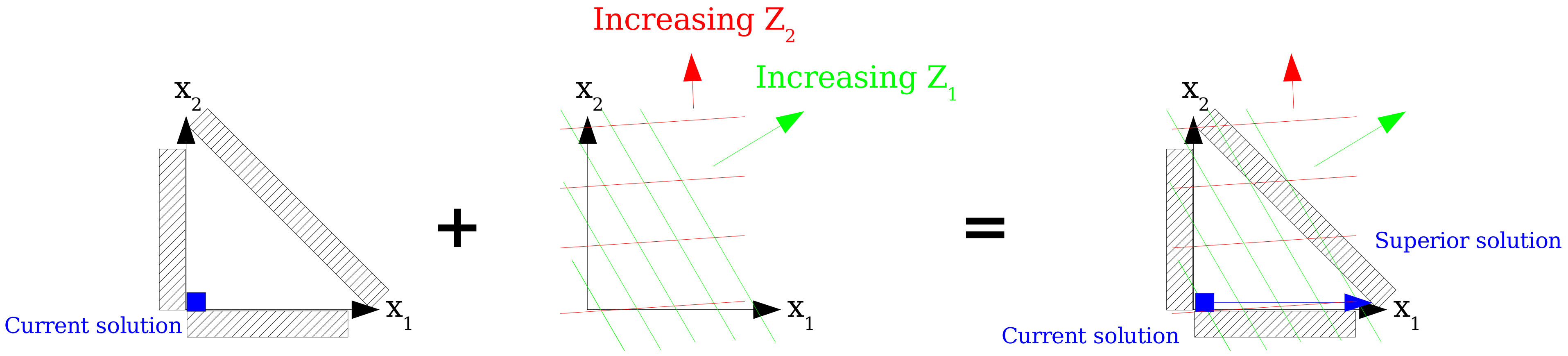
x_{NB2} will now become basic

Which currently basic variable will it replace? Answer comes in Step 12

Step 6

Is the current solution noninferior by a less obvious manner?

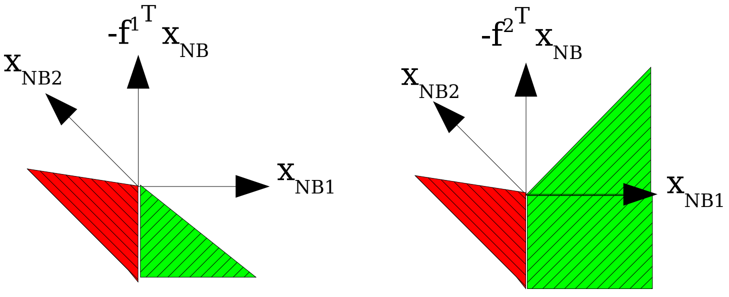
Solve the non-dominance problem



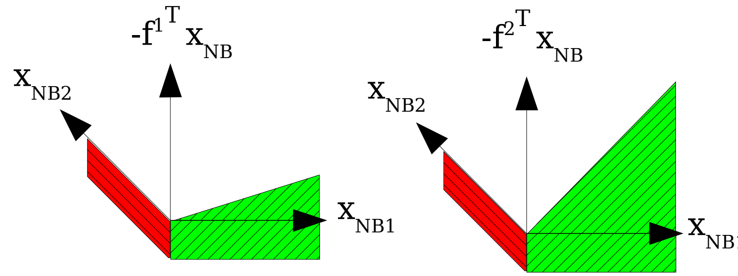
Step 8

$-\theta_j \vec{\mathbf{f}}_{\mathbf{j}} \geq -\theta_q \vec{\mathbf{f}}_{\mathbf{q}}$

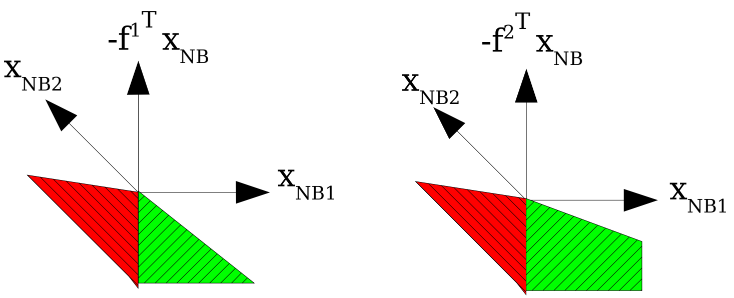
Is one search direction obviously better than all others? Even if it leads to a decrease in any of the objectives?
(Are the scaled gradients with respect to one nonbasic variable greater than the scaled gradients of any other nonbasic variable?)



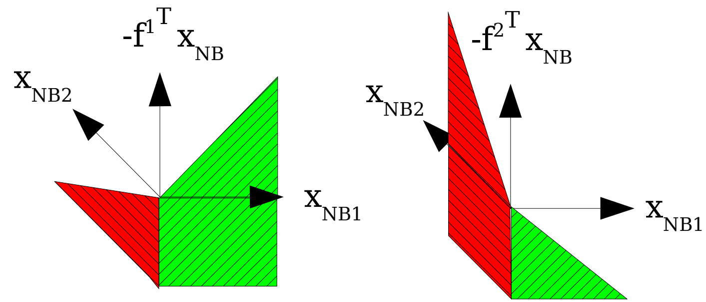
x_{NB1} is non-dominated



x_{NB1} is non-dominated



x_{NB1} is non-dominated

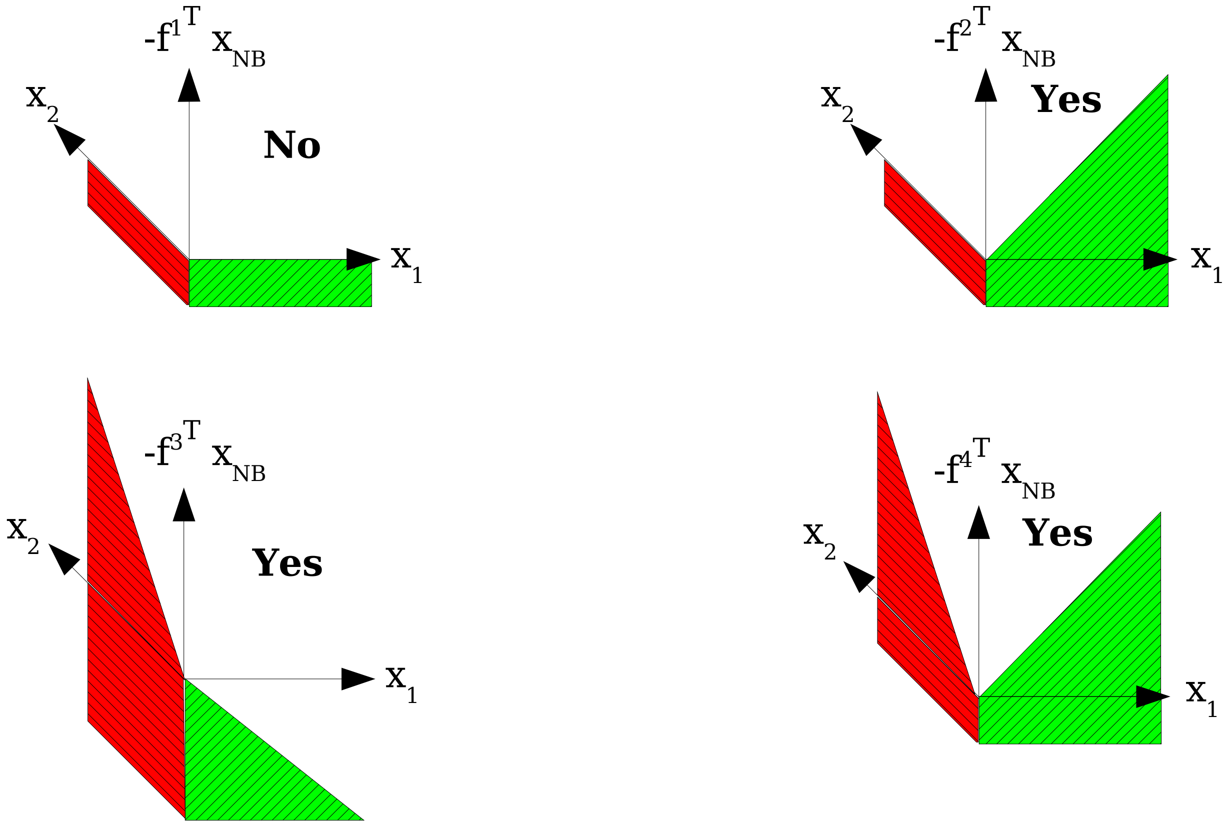


Neither variable/direction
is non-dominated

Step 9a

Any reduced costs for any objective not zero?

If so, it may lead to unexplored bases (to be checked in Step 12)

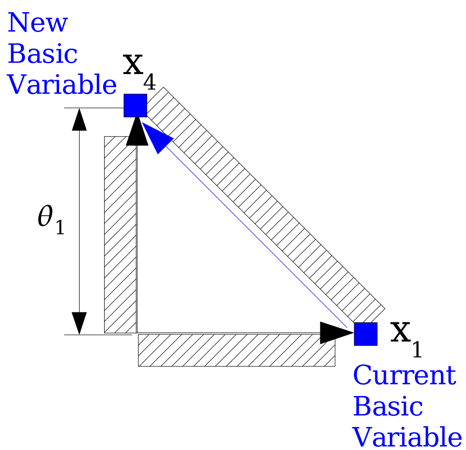


Step 12

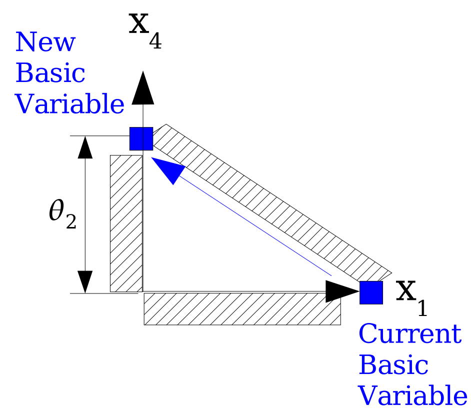
Would the introduction of a nonbasic variable lead to an unexplored basis?
Form the basis, and then check whether it is new.

Introduce the nonbasic variable x_4
and check each basic variables to determine whether the constraints impose a maximum increment for the incoming nonbasic variable

$$\theta_1 = \frac{x_1}{-\left\{\dfrac{dx_4}{dx_1}\right\}}$$



$$\theta_2 = \frac{x_1}{-\left\{\dfrac{dx_4}{dx_2}\right\}}$$



All of the thetas represent the maximum allowable step size before a currently basic variable becomes zero.
Therefore, choose the smallest maximum step size.

Step 13

Introduce a nonbasic variable and remove a basic variable to form a new basis

Introduce the nonbasic variable x_3 and remove the basic variable x_1