MAPL 699
Optimization and Equilibrium Problems, Spring 2001, Professor Gabriel

Theorem

Let $f : R^n \to R$ be a convex program and let $S \subseteq R^n$ be a convex set. Then, the convex program

$$
\textbf{(1)} \qquad \begin{aligned} &\min \ f(x) \\ &s.t \\ &x \in S \end{aligned}
$$

has either 0, 1, or an infinite number of solutions.

Proof

Consider the convex program

(2) $\min \ c_1 x \ s.t. \ x \geq b_1, x \leq b_2$.

<u>Zero solutions</u>

Let $c_1 = 1, b_1 = 0, b_2 = -1$, then (2) has no solutions since the feasible region is empty. (The same conclusion could be reached if we took $c_1 = -1, b_1 = 0$ and we removed the second inequality since the objective function would be unbounded.)

<u>One solution</u>

Let $c_1 = 1, b_1 = 0, b_2 = 1$, then (1) has exactly one solution, namely x=0.

<u>Infinite number of solutions</u>

Let x and y be two distinct solutions to (1) and let z=w x + (1-w) y where w is a scalar in (0,1). Then by definition of S being a convex set, z is in S. By the convexity of f we have $f(z) = f(wx + (1-w)y) \leq wf(x) + (1-w)f(y)$, which implies that z is also optimal given that x and y were. Hence, there are an infinite number of solutions.

QED