Theorem

$$\text{Sol}(NCP(F, l, u)) = \text{sol}(VI([l,u], F))$$

Proof

$$\text{sol}(VI([l,u], F)) \subseteq \text{sol}(NCP(F, l, u))$$

Let x^* be a solution to $VI([l,u], F)$, i.e.,

$x^* \in [l,u]$ or $l_i \le x_i \le u_i \ \forall i$ and

$$F(x^*)^T (y - x^*) \geq 0 \quad \forall y \in [l,u].$$

$$\left(\text{or} \ \sum_i F_i(x^*)(y_i - x_i) \geq 0 \quad \forall y \in [l,u]. \right)$$

Take $y = x^* + \varepsilon e_i$ for $\underline{x_i^* = l_i}$, e_i standard basis vector, $\varepsilon > 0$ small enough so that $x_i + \varepsilon e_i \le u_i$, ok since $l_i < u_i$

$\Rightarrow y \in [l,u]$.

& $\Rightarrow F(x^*)^T (x^* + \varepsilon e_i - x^*) \geq 0$

$\Rightarrow \underline{F_i(x^*) \geq 0}$

Take $y = x^* - \varepsilon e_i$ for $\underline{x_i^* = u_i}$, ε small enough so that $x_i - \varepsilon e_i \geq l_i$, ok since $l_i < u_i$

$\Rightarrow y \in [l,u]$

&

$\Rightarrow F(x^*)^T (x^* - \varepsilon e_i - x^*) \geq 0$

$\Rightarrow \underline{F_i(x^*) \le 0}$

Take $y = x^* + \varepsilon e_i$ for $\underline{l_i < x_i^* < u_i}$, ε small enough so that $x_i + \varepsilon e_i \in [l_i, u_i]$ ok since $l_i < u_i$

$\Rightarrow y \in [l,u]$

&

$\Rightarrow F(x^*)^T (x^* + \varepsilon e_i - x^*) \geq 0$

$\Rightarrow F_i(x^*) \geq 0$ same idea for $y = x^* - \varepsilon e_i$ with ε small enough so that $x_i - \varepsilon e_i \in [l_i, u_i]$.

$\Rightarrow F_i(x^*) \le 0$

or $\underline{F_i(x^*) = 0}$

$\therefore \quad x^* \in Sol(NCP(F, l, u))$

$Sol(NCP(F, l, u)) \subseteq sol(VI([l,u], F))$

let x^* be a solution to $NCP(F, l, u)$ i.e.,

$$F_i(x^*) \geq 0 \quad , \quad x_i^* = l_i$$
$$F_i(x^*) = 0 \quad , \quad l_i < x_i^* < u_i$$
$$F_i(x^*) \leq 0 \quad , \quad x_i^* = u_i$$

But then we have

for $x_i^* = l_i$ $\qquad \underbrace{F_i(x^*)}_{\geq 0} \underbrace{(y_i - x_i^*)}_{\geq 0 \text{ since } y_i \in [l_i, u_i],\ x_i^* = l_i} \geq 0$

for $x_i^* \in (l_i, u_i)$ $\qquad \underbrace{F_i(x^*)}_{= 0} \underbrace{(y_i - x_i^*)}_{?} \geq 0$

for $x_i^* = u_i$ $\qquad \underbrace{F_i(x^*)}_{\leq 0} \underbrace{(y_i - x_i^*)}_{\leq 0} \geq 0$ since $y_i \in [l_i, u_i]$, $x_i^* = u_i$

$$\Rightarrow 0 \leq \sum_{i:\, x_i^* = l_i} F_i(x^*)(y_i - x_i^*) + \sum_{i:\, x_i \in (l_i, u_i)} F_i(x^*)(y_i - x_i^*) + \sum_{i:\, x_i^* = u_i} F_i(x^*)(y_i - x_i^*)$$

$$= F(x^*)^T (y - x^*)$$

$\therefore \quad x^* \in sol(VI([l,u], F)) \quad \boxed{}$